Research Statement

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My primary area of research is Functional Analysis. I have worked on long standing problems on geometry of Banach spaces. One of the geometric consequences of Hahn Banach Theorem implies that for a closed bounded convex set C in a Banach Space X, a point P outside, can be separated from C by a hyperplane. The question whether this separation can be done in terms of intersection or union of balls has intrigued mathematicians in this area. In my thesis, I have answered this question in varying degree in terms of “nice” points (like extreme points) in the dual unit ball. These properties are closely related to the Radon Nikodym Property (RNP) in Banach Spaces. I have studied a large class of these ball intersection properties and obtained equivalent condition on spaces with such properties to also have RNP. I have also studied various stability results of these properties and the behavior of these properties in spaces of operators. I have been able to obtain necessary and sufficient condition among different version of ball separation properties as well. I have constructed examples to distinguish between all such existing notions.

These Ball separation properties can be classified into two types, hereditary and non-hereditary. In my work, Asymptotic Norming Properties (ANP) of three types along with their w*-versions in the hereditary class, are studied. A new kind of ANP is also introduced which has pleasant geometric properties. Several stability results of ANP's are established. For the non-hereditary class, I examined Nicely smooth spaces, Property II and Ball Generated property (BGP) and obtained a variety of stability results. The next step was to study of all these properties in C (K, X)-spaces where K is a compact Hausdorff space and X is a Banach space. I obtained a complete characterization of C (K, X) spaces with these properties. These properties are also studied in L (X, Y) the space of bounded operators between Banach spaces X and Y. After the completion of doctoral work, I have been studying the Ball Generated Property (BGP) on Banach spaces and nicely smooth spaces extensively. I introduced a generalization of nicely smooth spaces. I have obtained new and interesting results on BGP in the context of operator spaces and tensor product spaces.

One of my current research area is small combination of slices (SCS) in Banach spaces. An important geometric property in its own right, SCS is also closely linked to other geometric properties in Banach spaces namely the Radon Nikodym property (RNP) and the Krein Milman Property (KMP). RNP implies KMP is a well-known result. Whether KMP implied RNP was a long standing open question. Eventually it was proved that a Banach space has RNP if and only if it has KMP and all closed bounded convex set has SCS. In my joint wok with

Professor T.S.S.R.K. Rao, we introduce the notion of Ball-SCS (BSCS), which can be seen as a generalization of dentability, an important geometric property of Banach spaces, in terms of SCS. We study certain stability results for the BSCS leading to a discussion on BSCSP in the context of ideals of Banach spaces. We prove that the BSCSP can be lifted from an M-ideal to the whole Banach Space. We also prove similar results for strict ideals and U-subspaces of a Banach space. We study certain stability results for SCS leading to a discussion on SCS in the context of ideals of Banach Spaces and spaces of operators. For a compact Hausdorff space K and a Banach space X, we study SCS in C (K, X) spaces and L (X, Y) for Y= C(K). We also study SCS in the context of tensor products of Banach spaces.

Along with my doctoral student Susmita Seal, we further extend the ideal of BSCSP to two other concepts, the Ball Dentable Property (BDP) i.e., the closed unit ball has slices of arbitrarily small diameter and Ball Huskable property (BHP)namely the closed unit ball has weakly open sets of arbitrarily small diameter. We show that BDP implies BHP which in turn implies BSCSP and none of the implications can be reversed. We now call these properties small diameter properties. These properties can be looked upon as local versions (to the closed unit ball) of Radon Nikodym Property, Point of Continuity Property and Strongly Regularity, the three important notions in geometry of Banach spaces. We prove stability results of these small diameter properties in the context of M-summands, P-summands and Lebesgue Bochner spaces. We further use these results to examine situations in the context of ideals in Banach Spaces. We study these problems in the context of operator spaces. We explore stability of these properties in the context of three space property. We connect the small diameter properties of Banach spaces to the differentiability of the norm. The full list of references of these results are included in the curriculum vitae.

Open Questions:

i) How can several density properties of SCS be realized as a ball separation property?

ii) What stability results will hold for these properties in the context of injective and projective tensor product spaces?

iii) How can we characterize these small diameter properties in terms of SCS points, denting points and point of continuity points in of the closed unit ball?

iv) D(2P)-properties is a recent topic which generated a lot of interest in the study of Banach spaces, it is known that Banach spaces with Daugavet properties have these properties. It is also know that Daugavet properties do not have RNP. In fact, one can conclude easily that Banach Spaces with Daugavet properties do not have SCS. But there are examples of spaces with D-2P which has SCS. So it will be interesting to examine where all ball separation stand in the context of D-2P properties.

v) Recently it was proved that every Banach space containing isomorphic copies of c_0 can be equivalently renormed so that **every** nonempty relatively weakly open subset of its unit ball has diameter 2 (D-2P property), however, the unit ball still contains **slices of arbitrarily small diameter i.e., has SCS.** There are several versions of D-2P properties, and it will be interesting to explore the relations between several D-2P properties and the several density properties that arise from the SCS, denting and PC points.

Operator Theory and Linear Hahn Banach Extension Operator

Apart from Banach spaces, I am also actively working on Operator theory. The notion of linear Hahn-Banach extension operator is a topic of interest for long time. Different versions of this notion have been studied in the context of non-separable reflexive Banach spaces. Subsequently, the existence of linear Hahn Banach extension operators was proved via interspersing subspaces in a purely Banach space theoretic set up. In my joint work professor with Professor A. I. Singh, we study similar questions in the context of Banach modules, in particular, Hilbert modules and Banach algebras. We study the conditions on Banach and Hilbert modules for the existence of Hahn Banach Extension Type operator. We first give examples to show that a generalization of the above result is not possible for all Banach modules. Consequently, we investigate situations where Hahn Banach type extensions exist as module homomorphisms. Finally, we look at interspersing Banach submodules. Our study includes in particular the space C(S) and conditions on it for the existence of Hahn Banach extension operators. We answer the question completely in the context of Banach Algebras of bounded operators on a Banach space considered as modules over itself. We first show that the result is true when the Banach space is reflexive. We give a counter example to show that the result fails in the non-reflexive case. In this case, we completely classify the interspersing submodules. The full list of references of these results are included in the curriculum vitae.

Open Questions:

As is evident from the above discussion the question of Hahn Banach extension operator for Module homomorphisms can only be examined in isolated special cases separately.

We plan to continue this work further for Banach modules endowed with interesting structural properties. Some relevant questions are:

i) What additional conditions can we impose on the Banach Module to get "plenty" of Linear Hahn Banach Extension Operator for module homomorphisms?

ii) What conditions can we impose on the Banach algebra so that we can "plenty" of Linear Hahn Banach Extension Operator on Banach Modules defined over the Banach Algebra endowed with those extra properties?

Adjoints of Operator in Banach Spaces: Apart from that I have also explored certain problems in operator theory like defining adjoint for operators in separable Banach spaces. This result is used to extend well-known theorems of von Neumann and Lax. We also partially solve an open problem on the existence of a Markushevich basis with unit norm and prove that all closed densely defined linear operators on a separable Banach space can be approximated by bounded operators. This last result extends a theorem of Kaufman for Hilbert spaces and allows us to define a new metric for closed densely defined linear operators on Banach spaces. As an application, we obtain a generalization of the Yosida approximator for semigroups of operators. This was a collaboration with Professor T. Gill. The full list of references of these results are included in the curriculum vitae.

Open Questions

We plan to generalize our results for non-separable Banach spaces.

P-adic Functional Analysis: The other operator theoretic results I have explored is in the context of p-adic Hilbert spaces. We have defined Hilbert Schimdt operators in p-adic Hilbert spaces and proved counterparts of many classical results in this context. Also, I have explored Schatten class and trace class operator in p-adic Hilbert spaces and proved many results echoing the classical theory. We also show that the Trace class operators in p-adic Hilbert spaces strictly contains the class of completely continuous operators. We give examples of Trace class operators which are not completely continuous in the context of p-adic Hilbert spaces. The full list of references of these results are included in the curriculum vitae.

Open Questions

i) What will be analogue compact operators in p-adic Hilbert spaces?

ii) How much classical operator theoretic results will be true here?

iii) What will be the analogue of spectral theorem in this context?

Spear vectors and Spear operators:

Most recently, I am part of a discussion group comprising mathematicians from University of Memphis, Virginia Tech University, and myself. The theory of spear vector and operators is an emerging topic in operator theory. The concepts of numerical range and numerical radius of operators, and numerical index of Banach spaces, played an important role in operator theory, particularly in the study and classification of operator algebras. These concepts of are also related to some of the well-known geometric properties of Banach spaces. This project is very recent, we are now trying to identify suitable problems in this area.